

On the anomalous torque applied to a rotating magnetized sphere in a vacuum

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Abstract

We analyze the torque applied to a rotating magnetized sphere in a vacuum. It is shown that for the correct determination of one of the torque's component the angular momentum of the electromagnetic field within the body should be taken into account.

1 Introduction

As is well known, the first model proposed to describe magnitospheres of pulsars – rotating neutron stars – was the simplest vacuum model [1, 2]. According to this model, which dates back to the classical paper by Deutsch [3], a neutron star can be viewed as a highly conducting magnetized solid sphere (with a radius R and a magnetic moment \mathbf{m}), rotating in the vacuum with an angular velocity Ω . The main power generation occurs due to magnetic-dipole¹ radiation, which decelerates the rotation and decreases the angle χ the spin axis z' and magnetic moment \mathbf{m} [4]. The projection of the braking torque on the spin axis is then expressed as

$$K_{z'} = -\frac{2}{3} \frac{m^2}{R^3} \left(\frac{\Omega R}{c} \right)^3 \sin^2 \chi, \quad (1)$$

and the total power $W_{\text{tot}} = -(\Omega \mathbf{K})$ is [5]

$$W_{\text{tot}} = \frac{2}{3} \frac{m^2 \Omega^4}{c^3} \sin^2 \chi. \quad (2)$$

The time evolution of the angle χ is described by the projection of the torque on the x' axis lying in the plane

¹The electric quadrupole radiation due to charge redistribution within the sphere is supposed by the factor $(\Omega R/c)^4$.

$\mathbf{m}\Omega$ which therefore also rotates around the z' axis with the angular velocity Ω :

$$K_{x'} = \frac{2}{3} \frac{m^2}{R^3} \left(\frac{\Omega R}{c} \right)^3 \sin \chi \cos \chi. \quad (3)$$

It is easy to verify that in this case, the braking torque \mathbf{K} is perpendicular to the magnetic dipole \mathbf{m} . Therefore, according to Euler's equations, the projection of the angular velocity on this axis must be conserved [4]

$$\Omega \cos \chi = \text{const}. \quad (4)$$

As we see, the characteristic time of evolution of the inclination angle χ and the angular velocity Ω are the same.

Later, however, it was found that if the neutron star magnitosphere is filled with plasma that screens the longitudinal electric field (parallel to the magnetic field), then the magnetospheric plasma fully suppresses the magneto-dipole radiation [6, 7]. Energy losses must in this case be related to the action of the Ampere force caused by the surface currents that close longitudinal currents in the pulsar magnitosphere; in the case of zero longitudinal currents, the total energy loss is zero.

Presently, this statement, which had been aggressively debated for many years after the publication of paper [6] in 1983, can be considered to have been proved. For example, a numerical solution of the inclined rotator, obtained by Spitkovsky [8] in the force-free approximation, does not contain the magneto-dipole wave [9]. We stress that, as shown below, the braking of a magnetized sphere rotating in the vacuum is also due to the surface currents [10, 11], but in this case these are purely vortex surface currents without sources or sinks.

Here, we do not discuss the model of a magnetosphere filled with plasma, but consider the apparently completely studied problem of a rotating magnetized sphere

in a vacuum. Even in the framework of this simple task, some problems remain open. In particular, there is no common opinion regarding the so-called anomalous torque, i.e., the one acting along y' axis perpendicular to the plane $\mathbf{m}\boldsymbol{\Omega}$ and leading not to regular decrease in inclination angle χ but to the precession of the spin axis. The name is due to the value of this torque,

$$K_{y'} = \xi \frac{m^2}{R^3} \left(\frac{\Omega R}{c} \right)^2 \sin \chi \cos \chi, \quad (5)$$

where ξ is a numerical coefficient of the order of unity, which turns out to be $(\Omega R/c)^{-1}$ times the braking torque $K_{z'}$. Here, different authors have obtained different values of ξ , namely $\xi = 2/3$ [2], $\xi = 1$ [4, 12], $\xi = 1/5$ [13], and $\xi = 3/5$ [14] (see also paper [15], in which, however, the electric field contribution was admittedly ignored). On the other hand, according to [10, 16], the anomalous torque is equal to zero ($\xi = 0$), and therefore no magnetized sphere precession should occur.

Clearly, such a situation, in which there is no full agreement on the solution of an apparently elementary problem, is curious. We recall that the anomalous torque applied to a neutron star causes its precession, and for a nonspherical star, this precession, superimposed on the deceleration of the radio pulsar rotation, should significantly affect the so-called braking index $n_{\text{br}} = \dot{\Omega}\Omega/\dot{\Omega}^2$ [14, 17, 18].

Thus, the problem considered is of both theoretical and purely practical interest. In this paper, we therefore try to clarify the situation as much we can, and show where the variety of results come from. As we see, different papers have in fact discussed different quantities, many of which cannot be treated as the torque applied to a magnetized sphere rotating in a vacuum. We perform the calculation independently in the framework of the so-called quasistationary formalism, which, as we see, allows obtaining the result in the fastest and most straightforward way.

2 Method of calculations

We first make several general comments. As noted in the Introduction, we are interested only in the anomalous torque applied to a magnetized sphere rotating in a vacuum. In addition, an important refinement is in order: as follows from the z -component of the equation of motion $d\mathbf{m}/dt = [\boldsymbol{\Omega} \times \mathbf{m}]$

$$\frac{d\mathbf{m}_z}{dt} = \Omega_x \mathbf{m}_y - \Omega_y \mathbf{m}_x, \quad (6)$$

the regular variation of $\mathbf{m}_z = \mathbf{m} \cos \chi$ related to magnetodipole energy losses is possible if there is a nonzero component of the angular velocity $\boldsymbol{\Omega}$ lying in the plane xy . However, as follows from the same formula, the value of Ω_{\perp} should be of the order of the inverse time of evolution of the angle χ . Therefore, as can be readily verified, this additional rotation can be ignored in the analysis of the anomalous torque.

Everywhere below, we assume the solid sphere to be ideally conducting, and hence the condition of magnetic field freezing holds everywhere within it:

$$\mathbf{E} + \beta_R \times \mathbf{B} = 0, \quad (7)$$

where $\beta_R = \boldsymbol{\Omega} \times \mathbf{r}/c$. Clearly, to determine the torque applied to the sphere due to the electromagnetic field, it is necessary to calculate the volume and surface currents and changes connected to the sphere rotation. As a result, forces applied to the sphere can be presented in the form

$$d\mathbf{F} = \rho_e \mathbf{E} dV + \frac{[\mathbf{j} \times \mathbf{B}]}{c} dV + \sigma_e \mathbf{E} dS + \frac{[\mathbf{I}_s \times \mathbf{B}]}{c} dS, \quad (8)$$

where the first and second pair of terms in the right-hand side respectively correspond to the bulk and surface effects. But if we assume that only corotation currents $\mathbf{j} = c\rho_e\beta_R$ are present within the body (which is our key assumption), then it is easy to verify that the bulk part of force (8) vanishes. Now, taking into account that on the sphere $\mathbf{r} = R\mathbf{n}$ and $dS = R^2 d\phi$, where $d\phi$ is the solid angle element, we obtain the total torque $\mathbf{K} = \int \mathbf{r} \times d\mathbf{F}$ in the form

$$\mathbf{K} = \frac{R^3}{4\pi} \int \left([\mathbf{n} \times \{\mathbf{B}\}] (\mathbf{B}\mathbf{n}) + [\mathbf{n} \times \mathbf{E}] (\{\mathbf{E}\} \mathbf{n}) \right) d\phi, \quad (9)$$

where the curly brackets denote field jumps on the sphere². Thus the calculation of the torque is reduced to determining the electromagnetic field inside and outside the sphere.

We next note that due to the linearity of Maxwell equations, all electromagnetic fields can be decomposed into axially symmetric components (with the magnetic axis parallel to the spin axis) and orthogonal components. The general solution has the form

$$A = A^{\parallel} \cos \chi + A^{\perp} \sin \chi, \quad (10)$$

where A is an arbitrary field component. Equation (5) suggests that only cross terms in which one of the component in the products $[\mathbf{n} \times \{\mathbf{B}\}] B_r$ and $[\mathbf{n} \times \mathbf{E}] \{E_r\}$ relates to the orthogonal component make a nonzero contribution to integral (9). For example, for a point-like magnetic dipole, the axially symmetric component coincides with the static magnetic field

$$\mathbf{B}^{\parallel} = \frac{3(\mathbf{m}\mathbf{n})\mathbf{n} - \mathbf{m}}{r^3}. \quad (11)$$

For an orthogonal rotator (and again in the case of a

²Here it is important that the field components outside the curly brackets are continuous on the sphere.

point-like dipole) these fields must have the form [3, 5]

$$B_r^\perp = \frac{m}{r^3} \sin \theta \operatorname{Re} \left(2 - 2i \frac{\Omega r}{c} \right) \times \exp \left(i \frac{\Omega r}{c} + i\varphi - i\Omega t \right), \quad (12)$$

$$B_\theta^\perp = \frac{m}{r^3} \cos \theta \operatorname{Re} \left(-1 + i \frac{\Omega r}{c} + \frac{\Omega^2 r^2}{c^2} \right) \times \exp \left(i \frac{\Omega r}{c} + i\varphi - i\Omega t \right), \quad (13)$$

$$B_\varphi^\perp = \frac{m}{r^3} \operatorname{Re} \left(-i - \frac{\Omega r}{c} + i \frac{\Omega^2 r^2}{c^2} \right) \times \exp \left(i \frac{\Omega r}{c} + i\varphi - i\Omega t \right), \quad (14)$$

$$E_r^\perp = 0, \quad (15)$$

$$E_\theta^\perp = \frac{m\Omega}{r^2 c} \operatorname{Re} \left(-1 + i \frac{\Omega r}{c} \right) \times \exp \left(i \frac{\Omega r}{c} + i\varphi - i\Omega t \right), \quad (16)$$

$$E_\varphi^\perp = \frac{m\Omega}{r^2 c} \cos \theta \operatorname{Re} \left(-i - \frac{\Omega r}{c} \right) \times \exp \left(i \frac{\Omega r}{c} + i\varphi - i\Omega t \right). \quad (17)$$

Finally, as in most papers, we consider the case of rather slow rotation, where the natural parameter

$$\varepsilon = \frac{\Omega R}{c} \quad (18)$$

is much smaller than unity; for most radio pulsars, $\varepsilon \approx 10^{-3}$ – 10^{-4} . Comparing expression (5) with general relation (9) for the torque \mathbf{K} , we then conclude that only the first two terms in the series expansion of the electric and magnetic fields in the parameter ε are needed in order to calculate the anomalous torque.

Therefore, in our opinion, the most convenient method of calculations is the so-called quasistationary formalism, which assumes that all fields depend on the azimuthal angle φ and time t only in the combination $\varphi - \Omega t$. In this case, the time derivatives can be substituted by spatial derivatives, and the Maxwell equations take the form [19]

$$\nabla \times (\mathbf{E} + \beta_R \times \mathbf{B}) = 0, \quad (19)$$

$$\nabla \times (\mathbf{B} - \beta_R \times \mathbf{E}) = \frac{4\pi}{c} \mathbf{j} - 4\pi \rho_e \beta_R. \quad (20)$$

Here, a clear advantage of the proposed method is revealed. Indeed, if we assume the corotation condition $\mathbf{j} = c\rho_e \beta_R$ to be valid inside the sphere, then the right-hand side of Eqn (20) vanishes both outside the sphere, where currents and charges are absent, and inside the

sphere. As a result, the following relations should hold both inside and outside the sphere:

$$\mathbf{E} + \beta_R \times \mathbf{B} = -\nabla \psi, \quad (21)$$

$$\mathbf{B} - \beta_R \times \mathbf{E} = \nabla h, \quad (22)$$

where $\psi(r, \theta, \varphi - \Omega t)$ and $h(r, \theta, \varphi - \Omega t)$ are scalar functions that can be found from the condition of continuity of the corresponding components of the electric and magnetic field and from the conditions $\nabla \mathbf{E} = 0$ and $\nabla \mathbf{B} = 0$ outside the sphere.

Moreover, the proposed method allows obtaining the desired result using a simple iteration procedure. Indeed, if the magnetic field $B^{(0)}$ is known in the zeroth order in the parameter $\varepsilon = \Omega R/c$, then, using Eqn (21), we can calculate the electric field $E^{(1)}$ in the first order in the parameter ε . Equation (22), in turn, allows finding the magnetic field $B^{(2)}$ in the second order. These two steps are sufficient to calculate the anomalous torque, which is proportional, as we have seen, to the square of the small parameter $\Omega R/c$.

Thus, the problem is reduced to finding two scalar functions $\psi^{(1)}$ and $h^{(2)}$, that fully determine the structure of electromagnetic fields to the required accuracy. Below, we omit indexes (1) and (2) in most cases.

3 Results

We first consider the simplest of a homogeneously magnetized solid sphere. This means that in the zeroth order in the parameter ε , the magnetic field is uniform inside the sphere and coincides with the field of a point-like dipole outside the sphere. Then the zeroth-order magnetic field components inside the sphere have the form

$$\begin{aligned} B_r^\perp &= \frac{2m}{R^3} \sin \theta \cos(\varphi - \Omega t), \\ B_\theta^\perp &= \frac{2m}{R^3} \cos \theta \cos(\varphi - \Omega t), \\ B_\varphi^\perp &= -\frac{2m}{R^3} \sin(\varphi - \Omega t), \end{aligned} \quad (23)$$

$$B_r^\parallel = \frac{2m}{R^3} \cos \theta, \quad B_\theta^\parallel = -\frac{2m}{R^3} \sin \theta, \quad B_\varphi^\parallel = 0. \quad (24)$$

Correspondingly, outside the sphere, we have

$$\begin{aligned} B_r^\perp &= \frac{2m}{r^3} \sin \theta \cos(\varphi - \Omega t), \\ B_\theta^\perp &= -\frac{m}{r^3} \cos \theta \cos(\varphi - \Omega t), \\ B_\varphi^\perp &= \frac{m}{r^3} \sin(\varphi - \Omega t), \end{aligned} \quad (25)$$

$$B_r^\parallel = \frac{2m}{r^3} \cos \theta, \quad B_\theta^\parallel = \frac{m}{r^3} \sin \theta, \quad B_\varphi^\parallel = 0. \quad (26)$$

We now turn to the first-order terms in the small parameter ε . We first note that in this order, the magnetic field is zero. This, unexpected at first glance, follows immediately from relations (12)–(14), where the exponential should be expanded in the Taylor series. The magnetic field cannot arise due to ∇h either, since in this order it would correspond to a monopole magnetic field.

As regard the electric field, by comparing Eqns (7) and (21), we obtain that the condition

$$\psi^{(\text{In})} = 0. \quad (27)$$

must always be satisfied inside the sphere. As a result, we have

$$E_r^{\perp(\text{In})} = \frac{2\mathfrak{m}}{R^3} \frac{\Omega r}{c} \sin \theta \cos \theta \cos(\varphi - \Omega t), \quad (28)$$

$$E_\theta^{\perp(\text{In})} = -\frac{2\mathfrak{m}}{R^3} \frac{\Omega r}{c} \sin^2 \theta \cos(\varphi - \Omega t), \quad E_\varphi^{\perp(\text{In})} = 0,$$

$$E_r^{\parallel(\text{In})} = -\frac{2\mathfrak{m}}{R^3} \frac{\Omega r}{c} \sin^2 \theta, \quad (29)$$

$$E_\theta^{\parallel(\text{In})} = -\frac{2\mathfrak{m}}{R^3} \frac{\Omega r}{c} \sin \theta \cos \theta, \quad E_\varphi^{\parallel(\text{In})} = 0.$$

We note that for the axially symmetric component, the divergence of the electric field is nonzero, which corresponds to a nonzero charge density inside the sphere:

$$\rho_{\text{GJ}} = -\frac{\Omega \mathbf{B}}{2\pi c}. \quad (30)$$

This ρ_{GJ} is referred to as the Goldreich-Julian charge density [20], named after the first to obtain this expression for neutron stars. For the orthogonal component, owing to the condition $\Omega \mathbf{B} = 0$, the volume charge density inside the sphere is zero.

On the other hand, outside the sphere, according to (21) with zero potential $\psi = 0$, the electric field must have the form

$$E_r^{\perp(\text{Out})} = -\frac{\mathfrak{m}}{r^3} \frac{\Omega r}{c} \sin \theta \cos \theta \cos(\varphi - \Omega t), \quad (31)$$

$$E_\theta^{\perp(\text{Out})} = -\frac{2\mathfrak{m}}{r^3} \frac{\Omega r}{c} \sin^2 \theta \cos(\varphi - \Omega t),$$

$$E_\varphi^{\perp(\text{Out})} = 0,$$

$$E_r^{\parallel(\text{Out})} = \frac{\mathfrak{m}}{r^3} \frac{\Omega r}{c} \sin^2 \theta, \quad (32)$$

$$E_\theta^{\parallel(\text{In})} = -\frac{2\mathfrak{m}}{r^3} \frac{\Omega r}{c} \sin \theta \cos \theta, \quad E_\varphi^{\parallel(\text{Out})} = 0.$$

In this case, however, it is easy to verify that the electric field divergence is nonzero. Therefore, to obtain the divergence-free electric field outside the sphere, where there are no charges or currents by definition, these expressions should be corrected using the functions ψ in

(21). It is straightforward to verify that the condition $\nabla \mathbf{E} = 0$ for the total field (as well as condition of the continuity of the tangential electric field component at the sphere $r = R$) is satisfied for the functions

$$\begin{aligned} \psi_0^\perp &= \frac{\mathfrak{m}}{r} \frac{\Omega}{c} \sin \theta \cos \theta \cos(\varphi - \Omega t) \\ &\quad - \frac{\mathfrak{m}}{r^3} \frac{\Omega R^2}{c} \sin \theta \cos \theta \cos(\varphi - \Omega t), \\ \psi_0^\parallel &= -\frac{\mathfrak{m}}{r} \frac{\Omega}{c} \sin^2 \theta + \frac{1}{3} \frac{\mathfrak{m}}{r^3} \frac{\Omega R^2}{c} (3 \cos^2 \theta - 1). \end{aligned} \quad (33)$$

Here and below, of course, we use the fact that singularities are absent at the sphere center and at infinity (which is also why we have chosen only increasing solutions inside the sphere and solutions decreasing at infinity outside it), and that the total charge of the sphere must be zero³.

Thus, the electric field outside the sphere takes the form

$$\begin{aligned} E_r^{\perp(\text{Out})} &= -\frac{3\mathfrak{m}}{r^4} \frac{\Omega R^2}{c} \sin \theta \cos \theta \cos(\varphi - \Omega t), \\ E_\theta^{\perp(\text{Out})} &= -\frac{\mathfrak{m}}{r^2} \frac{\Omega}{c} \cos(\varphi - \Omega t) \\ &\quad + \frac{\mathfrak{m}}{r^4} \frac{\Omega R^2}{c} (1 - 2 \sin^2 \theta) \cos(\varphi - \Omega t), \\ E_\varphi^{\perp(\text{Out})} &= \frac{\mathfrak{m}}{r^2} \frac{\Omega}{c} \cos \theta \sin(\varphi - \Omega t) \\ &\quad - \frac{\mathfrak{m}}{r^4} \frac{\Omega R^2}{c} \cos \theta \sin(\varphi - \Omega t), \end{aligned} \quad (34)$$

$$\begin{aligned} E_r^{\parallel(\text{Out})} &= -\frac{\mathfrak{m}}{r^4} \frac{\Omega R^2}{c} (3 \cos^2 \theta - 1), \\ E_\theta^{\parallel(\text{Out})} &= -\frac{2\mathfrak{m}}{r^4} \frac{\Omega R^2}{c} \sin \theta \cos \theta, \\ E_\varphi^{\parallel(\text{Out})} &= 0. \end{aligned} \quad (35)$$

It is easy to verify that the orthogonal component of the electric field outside the sphere is the sum of the magnetic dipole radiation field (15)–(17) and the quadrupole field of charges induced in the sphere. The longitudinal field contains only the static field of the quadrupole; naturally, this component does not generate electromagnetic waves. Finally, jumps of the radial electric field component, which determine the surface charges, are expressed as

$$\begin{aligned} \{E_r^\perp\} &= -5 \frac{\mathfrak{m}}{R^3} \frac{\Omega R}{c} \sin \theta \cos \theta \cos(\varphi - \Omega t), \\ \{E_r^\parallel\} &= \frac{\mathfrak{m}}{R^3} \frac{\Omega R}{c} (3 - 5 \cos^2 \theta). \end{aligned} \quad (36)$$

Here, the total charge of the shell is nonzero, and the opposite-sign charge related to Goldreich-Julian charge

³We stress that, as seen from (21), the potential ψ_0^\parallel is not the total electric potential of a static axially symmetric problem.

density (30) is uniformly distributed within the sphere volume.

We now determine the second-order fields in ε . Relation (9) suggests that only the magnetic field is relevant here. Indeed, only the magnetic field that appears in products with zeroth-order magnetic field contributes to the anomalous torque. Meantime, the second-order electric field would contribute to only third-order terms in ε . However, as we can again see directly from relations (15)–(17), the electric field in this order simply vanishes:

$$\mathbf{E}^{(2)} = 0. \quad (37)$$

The second-order magnetic field can be calculated from the first-order electric field using Eqn (22). Using the same procedure as for electric fields, it is straightforward to find the compensating potentials h , that is needed for the condition $\nabla \mathbf{B} = 0$ to be satisfied. As a result, inside the sphere, we obtain

$$\begin{aligned} h^{\perp(\text{In})} &= -\frac{3}{5} \frac{\mathbf{m}}{R^3} \frac{\Omega^2 r^3}{c^2} \sin \theta \cos(\varphi - \Omega t), \\ h^{\parallel(\text{In})} &= 0. \end{aligned} \quad (38)$$

Correspondingly, outside the sphere,

$$\begin{aligned} h^{\perp(\text{Out})} &= \frac{\mathbf{m}}{2} \frac{\Omega^2}{c^2} \sin \theta \cos(\varphi - \Omega t) \\ &\quad - \frac{\mathbf{m}}{r^2} \frac{\Omega^2 R^2}{c^2} \sin \theta \cos 2\theta \cos(\varphi - \Omega t), \\ h^{\parallel(\text{Out})} &= \frac{\mathbf{m}}{2} \frac{\Omega^2}{c^2} \cos \theta + \frac{\mathbf{m}}{r^2} \frac{\Omega^2 R^2}{c^2} \cos \theta \sin^2 \theta. \end{aligned} \quad (39)$$

Therefore, inside the sphere the second-order magnetic field can be written as

$$\begin{aligned} B_r^{\perp(\text{In2})} &= \frac{\mathbf{m}}{R^3} \frac{\Omega^2 r^2}{c^2} \sin \theta \left(2 \sin^2 \theta - \frac{9}{5} \right) \cos(\varphi - \Omega t), \\ B_{\theta}^{\perp(\text{In2})} &= \frac{\mathbf{m}}{R^3} \frac{\Omega^2 r^2}{c^2} \cos \theta \left(2 \sin^2 \theta - \frac{3}{5} \right) \cos(\varphi - \Omega t), \\ B_{\varphi}^{\perp(\text{In2})} &= \frac{3}{5} \frac{\mathbf{m}}{R^3} \frac{\Omega^2 r^2}{c^2} \sin(\varphi - \Omega t), \end{aligned} \quad (40)$$

$$\begin{aligned} B_r^{\parallel(\text{In2})} &= \frac{2\mathbf{m}}{R^3} \frac{\Omega^2 r^2}{c^2} \sin^2 \theta \cos \theta, \\ B_{\theta}^{\parallel(\text{In2})} &= -\frac{2\mathbf{m}}{R^3} \frac{\Omega^2 r^2}{c^2} \sin^3 \theta, \\ B_{\varphi}^{\parallel(\text{In2})} &= 0. \end{aligned} \quad (41)$$

Correspondingly, outside the sphere we obtain

$$\begin{aligned} B_r^{\perp(\text{Out2})} &= \frac{\mathbf{m}}{r} \frac{\Omega^2}{c^2} \sin \theta \cos(\varphi - \Omega t) \\ &\quad + \frac{\mathbf{m}}{r^3} \frac{\Omega^2 R^2}{c^2} \sin \theta \left(4 \sin^2 \theta - \frac{13}{5} \right) \cos(\varphi - \Omega t), \\ B_{\theta}^{\perp(\text{Out2})} &= \frac{1}{2} \frac{\mathbf{m}}{r} \frac{\Omega^2}{c^2} \cos \theta \cos(\varphi - \Omega t) \\ &\quad + \frac{\mathbf{m}}{r^3} \frac{\Omega^2 R^2}{c^2} \cos \theta \left(-6 \sin^2 \theta + \frac{4}{5} \right) \cos(\varphi - \Omega t), \\ B_{\varphi}^{\perp(\text{Out2})} &= -\frac{1}{2} \frac{\mathbf{m}}{r} \frac{\Omega^2}{c^2} \sin(\varphi - \Omega t) \\ &\quad + \frac{\mathbf{m}}{r^3} \frac{\Omega^2 R^2}{c^2} \left(\sin^2 \theta - \frac{4}{5} \right) \sin(\varphi - \Omega t), \end{aligned} \quad (42)$$

$$\begin{aligned} B_r^{\parallel(\text{Out2})} &= \frac{4}{5} \frac{\mathbf{m}}{r^3} \frac{\Omega^2 R^2}{c^2} \cos \theta, \\ B_{\theta}^{\parallel(\text{Out2})} &= \frac{2}{5} \frac{\mathbf{m}}{r^3} \frac{\Omega^2 R^2}{c^2} \sin \theta, \\ B_{\varphi}^{\parallel(\text{Out2})} &= 0. \end{aligned} \quad (43)$$

It is easy to verify that the first terms in the orthogonal component (42) exactly coincide with the fields of a rotating magnetic dipole, which are proportional to r^{-1} ; to show this, it is again necessary to expand the exponential the relations (12)–(14). The second terms correspond to radiation fields of quadrupole radiation. As we see, the method we use indeed allows exactly reproducing the known results through the second order in ε . As regards the parallel component (43), the second-order magnetic field is simply the field of a magnetic dipole equal to $(2/5)\varepsilon^2$ times the magnetic dipole of the sphere \mathbf{m} . This field is generated by the circle corotation current $\mathbf{j} = \rho_e[\boldsymbol{\Omega} \times \mathbf{r}]$.

The above equations, however, do not yet solve the problem⁴. The potentials $h^{(2)}$ are determined up to free harmonic functions, which are solutions of the Laplace equation,

$$h^{(\text{In})} = \sum_{l=0}^{\infty} \sum_{m=-l}^l f_l^m r^l Y_l^m(\theta, \varphi), \quad (44)$$

$$h^{(\text{Out})} = \sum_{l=0}^{\infty} \sum_{m=-l}^l f_l^m r^{-l-1} Y_l^m(\theta, \varphi), \quad (45)$$

where $Y_l^m(\theta, \varphi)$ are spherical functions. Naturally, here again, only solutions increasing with r are chosen inside the sphere, and decreasing solutions are chosen outside it. This case, as we see, is different from that of the first-order electric field because for the potential inside the sphere, the condition $\psi = 0$ was chosen. The continuity

⁴This is already seen from the fact that for the normal component, the continuity condition on the sphere is not satisfied here.

of the tangential component caused additional harmonic fields to be also absent for $r > R$.

As is easy to verify, the potential $h^{(2)}$ can contain only those spherical functions that correspond to the angular distribution of the volume charges currents; therefore, for the orthogonal component, we finally obtain

$$\begin{aligned} h^{\perp(\text{In})} &= a^{\perp(\text{In})} \frac{\mathbf{m}}{R^3} \frac{\Omega^2 R^2}{c^2} r \hat{Y}_1^1(\theta, \varphi) \\ &+ b^{\perp(\text{In})} \frac{\mathbf{m}}{R^5} \frac{\Omega^2 R^2}{c^2} r^3 \hat{Y}_3^1(\theta, \varphi), \\ h^{\perp(\text{Out})} &= a^{\perp(\text{Out})} \frac{\mathbf{m}}{r^2} \frac{\Omega^2 R^2}{c^2} \hat{Y}_1^1(\theta, \varphi) \\ &+ b^{\perp(\text{Out})} \frac{\mathbf{m} R^2}{r^4} \frac{\Omega^2 R^2}{c^2} \hat{Y}_3^1(\theta, \varphi). \end{aligned} \quad (46)$$

Correspondingly, for the axially symmetric component, we have

$$\begin{aligned} h^{\parallel(\text{In})} &= a^{\parallel(\text{In})} \frac{\mathbf{m}}{R^3} \frac{\Omega^2 R^2}{c^2} r \hat{Y}_1^0(\theta, \varphi) \\ &+ b^{\parallel(\text{In})} \frac{\mathbf{m}}{R^5} \frac{\Omega^2 R^2}{c^2} r^3 \hat{Y}_3^0(\theta, \varphi), \\ h^{\parallel(\text{Out})} &= a^{\parallel(\text{Out})} \frac{\mathbf{m}}{r^2} \frac{\Omega^2 R^2}{c^2} \hat{Y}_1^0(\theta, \varphi) \\ &+ b^{\parallel(\text{Out})} \frac{\mathbf{m} R^2}{r^4} \frac{\Omega^2 R^2}{c^2} \hat{Y}_3^0(\theta, \varphi). \end{aligned} \quad (47)$$

For simplicity, we here use the "nonnormalized" spherical functions

$$\begin{aligned} \hat{Y}_1^0(\theta, \varphi) &= \cos \theta, \\ \hat{Y}_3^0(\theta, \varphi) &= 5 \cos^3 \theta - 3 \cos \theta, \\ \hat{Y}_1^1(\theta, \varphi) &= \sin \theta \cos \varphi, \\ \hat{Y}_3^1(\theta, \varphi) &= (5 \sin^3 \theta - 4 \sin \theta) \cos \varphi. \end{aligned} \quad (48)$$

Thus, the problem is reduced to determining eight coefficients (a and b with the various indexes), which are to be found from the normal-component continuity on the sphere surface. We hence obtain the following relations between the coefficients:

$$\begin{aligned} a^{\perp(\text{In})} &= -2a^{\perp(\text{Out})} + \frac{9}{5}, \\ b^{\perp(\text{In})} &= -\frac{4}{3}b^{\perp(\text{Out})} + \frac{2}{15}, \\ a^{\parallel(\text{In})} &= -2a^{\parallel(\text{Out})}, \\ b^{\parallel(\text{In})} &= -\frac{4}{3}b^{\parallel(\text{Out})} + \frac{2}{15}. \end{aligned} \quad (49)$$

As we see, relations (49) are insufficient to find all eight unknown coefficients, however. Indeed, to the fields considered in this order, which arise due to rotation of the sphere, we can add fields that are formally of the order ε^2 , but are not related to the rotation itself. Such fields can arise due to additional surface currents, not caused

by the sphere rotation, which are ε^2 times the surface currents generating the zeroth-order magnetic field.

The additional fields arising due to potentials (46)–(47) also contribute to the anomalous torque, and we cannot drop them in the full solution. Remarkably, however, the anomalous torque itself is independent of the choice of free coefficient.

Indeed, four such free coefficients can be taken to be $(a, b)^{\perp(\text{Out})}$ and $(a, b)^{\parallel(\text{Out})}$, which describe harmonic fields outside the sphere. Direct integration of the corresponding components in the general expression (9) shows that the anomalous torque is indeed independent of $(a, b)^{\perp(\text{In})}$ and $(a, b)^{\parallel(\text{Out})}$ because of relations (49). Just this must be the case, because if their contribution were nonzero, the contribution from the zeroth-ordered term $[\mathbf{n} \times \{\mathbf{B}^{(0)}\}]_y, B_r^{(0)}$ would be nonzero, which is also related to free fields described by harmonic functions. On the other hand, as can be seen from relations (49), if all $(a, b)^{\perp(\text{Out})}$ and $(a, b)^{\parallel(\text{Out})}$ are set equal to zero, some of the coefficients $(a, b)^{\perp(\text{In})}$ and $(a, b)^{\parallel(\text{In})}$ become nonzero and would therefore also contribute to K_y .

To uniquely determine the solution, we again assume that the second-order surface currents are solely due to rotation of the surface charge σ_e :

$$I_\varphi = \sigma_e \Omega R \sin \theta, \quad (50)$$

$$I_\theta = 0. \quad (51)$$

Conditions (50) and (51) yield additional relations needed to completely determine the coefficients:

$$\begin{aligned} a^{\perp(\text{Out})} &= \frac{7}{30}, & a^{\perp(\text{In})} &= \frac{4}{3}, \\ b^{\perp(\text{Out})} &= \frac{1}{7}, & b^{\perp(\text{In})} &= -\frac{2}{35}, \\ a^{\parallel(\text{Out})} &= 0, & a^{\parallel(\text{In})} &= 0, \\ b^{\parallel(\text{Out})} &= \frac{1}{7}, & b^{\parallel(\text{In})} &= -\frac{2}{35}. \end{aligned} \quad (52)$$

Importantly, the Deutsch solution [3] corresponds to a somewhat different problem setup. In [3] it was assumed that the normal magnetic field component on the sphere does not contain corrections of the order ε^2 at all. As can be easily verified, this solution corresponds to choosing the constants as

$$\begin{aligned} a^{\perp(\text{Out})} &= -\frac{4}{5}, & b^{\perp(\text{Out})} &= \frac{1}{5}, \\ a^{\parallel(\text{Out})} &= \frac{2}{5}, & b^{\parallel(\text{Out})} &= 0, \end{aligned} \quad (53)$$

whence

$$\begin{aligned} a^{\perp(\text{In})} &= \frac{1}{5}, & b^{\perp(\text{In})} &= -\frac{2}{15}, \\ a^{\parallel(\text{In})} &= -\frac{4}{5}, & b^{\parallel(\text{In})} &= \frac{2}{15}. \end{aligned} \quad (54)$$

Thus, in our setting, the Deutsch solution is the one for a rotation dipole with specially adjusted additional

sources of small dipole and octupole fields such that they compensate the the normal component of the magnetic field of the order ε^2 on the sphere. The value of the anomalous torque, as shown above, is independent of this choice.

We now turn to calculating the anomalous torque itself, Eqn (9), which can be represented in the form

$$K_{y'} = \frac{R^3}{4\pi} \int \left([\mathbf{n} \times \{\mathbf{B}^{(2)}\}]_{y'} B_r^{(0)} + [\mathbf{n} \times \{\mathbf{B}^{(0)}\}]_{y'} B_r^{(2)} + [\mathbf{n} \times \mathbf{E}^{(1)}]_{y'} \{E_r^{(1)}\} \right) d\Omega \quad (55)$$

Formula (55) can be simplified. Indeed, because the second-order surface current $I_\varphi = \sigma_e \Omega R \sin \theta$ is determined only by the surface charge associated with the jump of the first-order electric field, we find

$$\begin{aligned} \{B_\theta^{(2)}\} &= \frac{\Omega R}{c} \sin \theta \{E_r^{(1)}\}, \\ \{B_\varphi^{(2)}\} &= 0. \end{aligned} \quad (56)$$

Using (21), we can also write

$$E_\theta^{(1)} = -\frac{\Omega R}{c} \sin \theta B_r^{(0)} \quad (57)$$

(this component is continuous on the sphere, and therefore we can set $\psi = 0$), and hence the first and third terms in (55), as can readily be verified, cancel each other, and as a result we obtain

$$K_{y'} = \frac{R^3}{4\pi} \int [\mathbf{n} \times \{\mathbf{B}^{(0)}\}]_{y'} B_r^{(2)} d\Omega. \quad (58)$$

This means that in the absence of zeroth-order surface currents, the anomalous torque is zero.

After performing elementary integration, the total anomalous torque is found to be

$$K_{y'} = \frac{1}{3} \frac{\mathbf{m}^2}{R^3} \left(\frac{\Omega R}{c} \right)^2 \sin \chi \cos \chi. \quad (59)$$

The contribution from the surface currents is here given by

$$K_{y'}^B = \frac{\mathbf{m}^2}{R^3} \left(\frac{\Omega R}{c} \right)^2 \sin \chi \cos \chi, \quad (60)$$

and contribution from the electric field (i.e., the torque due to surface charges) is

$$K_{y'}^E = -\frac{2}{3} \frac{\mathbf{m}^2}{R^3} \left(\frac{\Omega R}{c} \right)^2 \sin \chi \cos \chi. \quad (61)$$

As the second example, we consider a rotating hollow sphere. In other words, we assume that charges and currents, including those that generate the zeroth-order magnetic field, are localized in a thin spherical shell with $r = R$. It turns out that this problem does not require changing the fields that we found for the orthogonal

dipole. Indeed, as noted above Goldreich-Julian charge density (30) for a uniform "horizontal" magnetic field inside the sphere is zero. Therefore, at $r < R$ we can again set $\psi^\perp = 0$.

As regards the axially symmetric component, in order to ensure the condition $\rho_e = 0$ inside the sphere, the potential

$$\delta\psi^\parallel = -\frac{2}{3} \frac{\mathbf{m}}{R^3} \frac{\Omega r^2}{c}. \quad (62)$$

must be added to the obtained solution. As a result, only an additional radial electric field arises inside the sphere:

$$\delta E_r^\parallel = \frac{4}{3} \frac{\mathbf{m}}{R^3} \frac{\Omega r}{c}. \quad (63)$$

whereas the electric field outside the sphere does not change at all. Here, the electric field jump on the surface is expressed as

$$\{E_r^\parallel\} = \frac{5}{3} \frac{\mathbf{m}}{R^3} \frac{\Omega R}{c} (1 - 3 \cos^2 \theta). \quad (64)$$

As we see, the full charge of the shell is zero in this case.

As regards the second-order magnetic field, it can easily be verified that the additional electric field (63) gives rise to the additional potential

$$\delta h^\parallel = \frac{4}{15} \frac{\mathbf{m}}{R^3} \frac{\Omega^2}{c^2} r^3 \cos \theta. \quad (65)$$

As a result, the additional magnetic field inside the sphere, including the free fields, takes the form

$$\begin{aligned} \delta B_r^{\parallel(\text{In2})} &= -\frac{4}{5} \frac{\mathbf{m}}{R^3} \frac{\Omega^2 r^2}{c^2} \cos \theta + a \frac{\mathbf{m}}{R^3} \frac{\Omega^2 R^2}{c^2} \cos \theta, \\ \delta B_\theta^{\parallel(\text{In2})} &= \frac{8}{5} \frac{\mathbf{m}}{R^3} \frac{\Omega^2 r^2}{c^2} \sin \theta - a \frac{\mathbf{m}}{R^3} \frac{\Omega^2 R^2}{c^2} \sin \theta, \\ \delta B_\varphi^{\parallel(\text{In2})} &= 0. \end{aligned} \quad (66)$$

Correspondingly, outside the sphere we obtain

$$\begin{aligned} \delta B_r^{\parallel(\text{Out2})} &= -\left(\frac{4}{5} + 2a' \right) \frac{\mathbf{m}}{r^3} \frac{\Omega^2 R^2}{c^2} \cos \theta, \\ \delta B_\theta^{\parallel(\text{Out2})} &= -\left(\frac{2}{5} + a' \right) \frac{\mathbf{m}}{r^3} \frac{\Omega^2 R^2}{c^2} \sin \theta, \\ \delta B_\varphi^{\parallel(\text{Out2})} &= 0. \end{aligned} \quad (67)$$

The continuity of the magnetic field normal component and the corotation condition yield

$$a = -\frac{2}{9}, \quad a' = \frac{4}{9}. \quad (68)$$

Hence, the full anomalous torque finally becomes

$$K_{y'} = \frac{31}{45} \frac{\mathbf{m}^2}{R^3} \left(\frac{\Omega R}{c} \right)^2 \sin \chi \cos \chi. \quad (69)$$

Using a similar method, we can also solve the problem in which the uniform zeroth-order magnetic field in the parameter ε occupies only the inner spherical volume with a radius R_{in} . For intermediate region $R_{\text{in}} < r < R$ (where, as for $r < R_{\text{in}}$, the potential ψ is zero) and for the region outside the sphere, we assume that the zeroth-order magnetic field is that of a point-like dipole. For the harmonic functions we then need as many as 16 coefficients, because both increasing and decreasing solutions can be taken in the region $R_{\text{in}} < r < R$. Eventually, we find

$$K_{y'} = \left(\frac{8}{15} - \frac{1}{5} \frac{R}{R_{\text{in}}} \right) \frac{m^2}{R^3} \left(\frac{\Omega R}{c} \right)^2 \sin \chi \cos \chi. \quad (70)$$

We see that at $R_{\text{in}} = R$, we recover the previous value $\xi = 1/3$.

4 Discussion

We have shown that the anomalous torque applied to a rotating magnetized sphere in a vacuum is not zero in the general case, and its value depends on the structure of the internal electromagnetic field. In particular, we should accept the divergence of the anomalous torque as R_{in} tends to zero; however, this situation is unphysical and cannot be realized.

We first discuss the results obtained in the previous studies. Unfortunately, in [4, 12], only the final result $\xi = 1$, is presented, which coincides, however, with the braking torque (60) due to surface currents only. It cannot be ruled out that those papers simply ignored the contribution from the electric component in Eqn (55). Next, we note that there is no direct contradiction with the result in [16], where the magnetic field inside the sphere was assumed to be that of a point-like dipole; then, as follows from Eqn (58), the contribution from the sphere surface to the anomalous torque (which is the only quantity determined in [16]) should be zero. In other papers, as we show below, a quite different quantity was considered, which does not have the meaning of a anomalous torque.

Indeed, in virtually all papers discussed above, the anomalous torque was calculated as the momentum flux ($K_i = -\int \varepsilon_{ijk} r_j T_{kl} dS_l$) of the electromagnetic stress tensor T_{kl} using the formula

$$\mathbf{K}^M = \frac{1}{4\pi} \int_S \left([\mathbf{r} \times \mathbf{B}] (\mathbf{B} dS) + [\mathbf{r} \times \mathbf{E}] (\mathbf{E} dS) - \frac{1}{2} (\mathbf{E}^2 + \mathbf{B}^2) [\mathbf{r} \times d\mathbf{S}] \right). \quad (71)$$

When integration over the sphere (with $\mathbf{r} = R\mathbf{n}$ and $d\mathbf{S} = \mathbf{n} R^2 d\Omega$, where, again, $d\Omega$ is the solid angle element), we obtain

$$\mathbf{K}_{y'}^M = \frac{R^3}{4\pi} \int \left([\mathbf{n} \times \mathbf{B}]_{y'} (\mathbf{B} \mathbf{n}) + [\mathbf{n} \times \mathbf{E}]_{y'} (\mathbf{E} \mathbf{n}) \right) d\Omega. \quad (72)$$

Formula (72) differs from (9) in that there are no electric or magnetic field jumps on the sphere surface. If we substitute the values of the fields outside the sphere as found above in (72), then at $r = R + 0$ we obtain

$$\xi = \frac{3}{5}, \quad (73)$$

which is the result in [14] for the Deutsch solution. Here, it is very important that the value $\xi = 3/5$ is also independent of the choice of free coefficients.

However, it should be kept in mind that relation (72), which indeed can be found in many textbooks, is provided with important comments in Landau and Lifshits's *Electrodynamics of Continuous Media* [21]. This formula can be used only if the considered volume 'does not include charged bodies that are field sources'. Therefore, formula (72) can be used only when the flux of the electromagnetic stress tensor within the body is zero. However, the rotating spherical body inside which currents and charges are induced does not satisfy this condition, as we now show.

Indeed, the flux of the angular momentum vector of the electromagnetic field is related to the torque acting on matter by the electromagnetic field angular momentum conservation law [22]

$$\frac{d\mathbf{L}_{\text{field}}}{dt} + \mathbf{K}^M + \int [\mathbf{r} \times \mathbf{F}] dV = 0. \quad (74)$$

Here, $\mathbf{L}_{\text{field}}$ is the angular momentum of the electromagnetic field inside the volume V ,

$$\mathbf{L}_{\text{field}} = \int \frac{[\mathbf{r} \times [\mathbf{E} \times \mathbf{B}]]}{4\pi c} dV \quad (75)$$

\mathbf{K}^M is the field angular momentum flux through the surface bounding this volume, and $\mathbf{F} = \rho_e \mathbf{E} + \mathbf{j} \times \mathbf{B}/c$ is the Lorentz force. The last term in (74) plays the role of a source or a sink and is therefore responsible for the angular momentum transfer from the electromagnetic field to matter:

$$\frac{d\mathbf{L}_{\text{mat}}}{dt} = \int [\mathbf{r} \times \mathbf{F}] dV. \quad (76)$$

It is this term that pays the role of torque, and not \mathbf{K}^M , as was assumed in [14, 17].

Indeed, consider a sphere of a radius $r < R$ concentric with the body of study. With the explicit expressions for the zeroth-order magnetic field and the first-order electric field, it can be readily verified that for a uniformly magnetized spherical body, the time-dependent angular momentum component of $\mathbf{L}_{\text{field}}$ in (75) has the form

$$\mathbf{L}_{\text{field}} = \frac{4}{15} \frac{m^2}{R^6} \left(\frac{\Omega r^5}{c^2} \right) \sin \chi \cos \chi \mathbf{e}_{x'}. \quad (77)$$

As the radius r increases, $\mathbf{L}_{\text{field}}$ continuously increases, and hence the angular momentum flux \mathbf{K}^M related to rotation of this vector increases; it is discontinuous only on the sphere $r = R$. Because $\dot{\mathbf{e}}_{x'} = \Omega \mathbf{e}_{y'}$, the time

derivative $\dot{\mathbf{L}}_{\text{field}} = \Omega L_{\text{field}} \mathbf{e}_{y'}$ exactly corresponds to the angular momentum flux \mathbf{K}^M , calculated for the inner surface of the sphere $r = R - 0$ ($4/15 = 3/5 - 1/3$).

Thus we arrive at an important conclusion: the force balance taken into account in the second order in ε , the electromagnetic field angular momentum should also be taken into account. Therefore, part of the stresses due to the electromagnetic field should affect the angular momentum of the field itself, and only the remaining part should influence the interaction with the rotating body. According to Eqn(74), this implies that for the torque acting on the sphere, we should use the expression

$$\mathbf{K} \equiv \frac{d\mathbf{L}_{\text{mat}}}{dt} = \{\mathbf{K}^M\}. \quad (78)$$

Incidentally, the divergence $\sim R_{\text{in}}^{-1}$ arising in Eqn(70) now becomes clear. Indeed, the direct calculation of the total angular momentum of the electromagnetic field inside the body in this case yields

$$\mathbf{L}_{\text{field}} = \frac{m^2 \Omega}{c^2 R} \left(\frac{1}{15} + \frac{1}{5} \frac{R}{R_{\text{in}}} \right) \sin \chi \cos \chi \mathbf{e}_{x'}. \quad (79)$$

As we see, the angular momentum of the field contained inside the sphere of radius R at given value of \mathbf{m} diverges as $1/(5R_{\text{in}})$. Therefore, the scale of forces applied to the rotating spherical body must be of the same magnitude but with the opposite sign.

In conclusion, we stress that the method proposed here is inapplicable to the calculation of the torque that is responsible for magneto-dipole radiation. As can be seen from (1), this torque must be of the third order in ε . Therefore, to determine this torque, the magnetic field $B^{(3)}$ in the third order in ε must be known (the electric field, owing to condition (37) does contribute in the third order). These fields, involved in products with $B^{(0)}$, must lead to the required value of \mathbf{K} (1).

But as can be easily verified, the third-order magnetic field $B^{(3)}$ is simply a uniform field, whose value cannot be determined by our procedure [11]⁵. Fortunately, this uncertainty arises only at the next step of the expansion, because, as we have seen, the anomalous torque (9) is $(\Omega R/c)^{-1}$ times the braking torque directed against the spin axis. As a result, the procedure described above is applicable to the problem posed.

On the other hand, if we take the uniform third-order magnetic field from explicit expressions for a rotating point-like dipole (12)–(14)

$$\mathbf{B}^{(3)} = -\frac{2}{3} \frac{\mathbf{m}}{R^3} \left(\frac{\Omega R}{c} \right)^3 \mathbf{e}_{y'}, \quad (80)$$

then the direct calculation of the electromagnetic angular momentum flux \mathbf{K}^M at $r = R - 0$ yields $K_{x'} = 0$ and $K_{z'} = 0$. At $r = R + 0$ naturally, we return to expressions (1) and (3). This means that in the third order in

ε , the electromagnetic field angular momentum within the body is zero. Therefore, to this order, the torque applied to a rotating sphere can indeed be determined in terms of the surface integral that has no field jumps. By contrast, as follows from numerical simulations [23], the flux $K_{x'}$ at $r > R$ depends on the integration radius. This means that outside the body, the third-order electromagnetic field angular momentum is also nonzero.

5 Conclusion

The anomalous torque $K_{y'}$ acting on a rotating magnetized sphere in the general case is indeed nonzero. However, $K_{y'}$ depends on the internal structure of the fields because in the second order in the parameter ε the electromagnetic field angular momentum $\mathbf{L}_{\text{field}}$ in (75) must be taken into account in the balance of forces, and this angular momentum in turn depends on the internal electric field structure. The result obtained in the three examples considered in Section 4 are different because, having the same normal magnetic field component $B_r^{(0)}$ on the sphere, each case has a different internal electromagnetic field structure. This result in different angular momenta of the electromagnetic field. But, in the third order in ε , the electromagnetic field angular momentum inside the body is zero. Therefore, when calculating the torques $K_{x'}$ and $K_{z'}$, we can use the angular momentum flux K_i^M in Eqn (71), which does not have field jumps on the surface.

At last, following Archimedes, we can cry "Eureka!" Indeed, the measurement of the anomalous torque applied to a rotating spherical body allows determining its internal structure, which has no apparent manifestations in the outer regions. As we have shown here, lower-order electromagnetic fields outside a solid and hollow sphere must be the same, while the applied torques are different by a factor of more than two. This is not the first such example in electrodynamics, however. For instance, if a body has the so-called anapole moment ⁶

$$\mathbf{T} = \frac{1}{10c} \int [(\mathbf{j}\mathbf{r})\mathbf{r} - 2r^2\mathbf{j}]dV, \quad (81)$$

then, in the absence of rotation, the electromagnetic fields outside the body are exactly equal to zero. But in a nonuniform magnetic field, the torque $\mathbf{K} = [\mathbf{T} \times [\nabla \times \mathbf{B}]]$ acts on the body, and the rotation of the body is accompanied by electromagnetic radiation [24]. Taking the field angular momentum into account is also absolutely necessary in some other cases (see, e.g. [25, 26] and references therein).

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⁵Apparently, this is because the method we use here does not discriminate between the retarded and advanced potentials [19].

⁶An anapole moment can be modeled by a toroidal solenoid with a poloidal winding current I .

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